Locally 1-to-1 Maps and 2-to-1 Retractions

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Abstract

This paper considers the question of which continua are 2-to- 1 retracts of continua.

1 Introduction.

A 2-to-1 retract is a continuum that is the image of an exactly 2-to-1 retraction defined on a continuum.

Most continua are not 2-to-1 retracts, using the word "most" as R.H. Bing did, because the pseudoarc is not a 2-to-1 retract; in fact, no hereditarily indecomposable continuum can be a 2-to-1 retract [1]. Many continua are known not to be 2-to-1 retracts because they are not 2-to-1 images of continua at all. Continua in this category, excluding some that are hereditarily indecomposable, include dendrites, arc-like continua, treelike arc continua, continua whose every subcontinuum has an endpoint, and continua whose every subcontinuum has a cut point. On the other hand, if a continuum contains a subcontinuum that is not unicoherent then the continuum is a 2-to-1 retract [4]. (At the end of the paper we have a glossary with definitions of lesser known terms.) But the fact that identifies the most 2-to- 1 retracts is that every continuum that contains a 2-to-1 retract of a continuum is a 2-to-1 retract [3]. But note that a solenoid shrugs off both criteria: a solenoid

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is a 2-to-1 retract, but none of its proper subcontinua (all arcs) are 2-to-1 retracts, and a solenoid is hereditarily unicoherent. In Section 2 we show how to construct some 2-to-1 retracts, how to identify some 2-to-1 retracts and how to identify some continua that are not 2-to-1 retracts, all using the odd fact that if some continuum X maps into a continuum Y and the map has a restriction (called a *1-to-1 cover*) to an open proper subset of X that maps 1-to-1 onto $f(X)$, then Y is a 2-to-1 retract of some other continuum.

We show, in Section 3, that maps defined on arclike continua or on hereditarily decomposable continua, or simple maps defined on treelike continua, have images that are 2-to-1 retracts provided the map is not a homeomorphism but is locally 1-to-1 (called a *strictly locally 1-to-1* map). But note that the 2-to-1 retractions themselves cannot be locally 1-to-1 at any boundary point of the image. We conjecture that the adjective *simple* (meaning $|f^{-1}(y)| \leq 2$ for each y in the image of f) can be removed from the hypothesis in the treelike case.

In section 4 we consider decomposable continua in more detail and we show that if $X = A \cup B$ is a decomposable continuum and A and B are proper subcontinua of X, then X is a 2-to-1 retract iff (1) either A or B is a 2-to-1 retract or (2) their intersection is not connected. This takes care of the decomposable case unless A and B can not be evaluated.

To partly justify our exclusive consideration of 2-to-1 retracts, we prove in the last section that if a continuum is a 2-to-1 retract of a continuum then it is a k-to-1 retract of a continuum for each positive integer k .

2 Maps with 1-to-1 covers and 2-to-1 retracts

The following theorem makes clear the connection between open covers of maps (see introduction or glossary for definitions) and 2-to-1 retracts, and its corollaries make clear its usefulness.

Theorem 1 *The following are equivalent for the continuum Y:*

1. *Y is a 2-to-1 retract of a continuum.*

2. *There is a simple map with a 1-to-1 cover from a continuum into Y.*

3. There is a map with a 1-to-1 cover from a continuum into Y.

Proof. Suppose $r : X \to Y$ is a 2-to-1 retraction from a continuum X onto Y. Then r is a simple map and $U = X \setminus Y$ is an open proper subset of X that r maps 1-to-1 onto Y. Hence the first statement implies the second. And the second statement easily implies the third. Suppose f is a map from a continuum X onto Y, and U is an open proper subset of X such that f is 1-to-1 on U and $f(U) = Y$. Define $g : X \to Y \times [0, \infty)$ by $g(x) = (f(x), d(x, X \setminus U))$. Let $Y' = Y \times \{0\}$ and let $Z = Y' \cup g(X)$. Since $Y' \cap g(X) \neq \emptyset$, Z is a continuum. The 2-to-1 retraction of Z onto Y' is defined by $r((y, t)) = (y, 0)$. Since Y' is homeomorphic to Y, the third statement implies the first.

One can construct many examples of hereditarily unicoherent 2-to-1 retracts using the first two corollaries to Theorem 1. For a very simple example, identify two points from different composants of any indecomposable continuum and use Corollary 2. Or use Corollary 1 and identify two disjoint subcontinua from different composants along a continuous map between the subcontinua. And Corollaries 3 and 4, rather than constructing 2-to-1 retracts, describe ways to decide if a given continuum is a 2-to-1 retract.

Corollary 1 *Suppose X is a continuum, D and E are disjoint subcontinua of X, and h is a map from D into E. Then $Y = X/\{\{x, h^{-1}(x)\} | x \in E\}$ is a 2-to-1 retract of a continuum.*

Proof. Let U be $X \setminus D$; then the quotient map $p : X \to Y$ maps U 1-to-1 onto Y.

Corollary 2 *Suppose X is a continuum and p and q are two points of X. Then $X/\{p, q\}$ is a 2-to-1 retract of a continuum.*

Corollary 3 *Suppose Y is a continuum and K is a local cut continuum that is not a cut continuum, i.e. $Y \setminus K$ is connected but there is an open set U containing K such that $U \setminus K = A \cup B$, two nonempty separated sets, and K contains both a limit point of A and a limit point of B. Then Y is a 2-to-1 retract of a continuum.*

Proof. Construct a continuum X by adding to the connected set $Y\backslash K$ two disjoint copies of K, say K^1 and K^2, with K^1 attached to A in the same way that K was attached to A and with K^2 attached to B in the same way that K was attached to B. Let h be the homeomorphism from K^1 to K^2 such that for each point t in K, h takes the copy of t in K^1 to the copy of t in K^2. Then, by Corollary 1, $Y = X/\{\{x, h^{-1}(x)\}|x \in K^2\}$ is a 2-to-1 retract of a continuum.

Corollary 4 *If the continuum Y has a local cut point that is not a cut point, then Y is a 2-to-1 retract.*

3 Strictly locally 1-to-1 maps and 2-to-1 retracts

The next series of results are intended to demonstrate that the strictly locally 1-to-1 image of a continuum is frequently a 2-to-1 retract because it has a 1-to-1 cover. Later we have two examples that demonstrate the sort of complexity that a continuum might have in order for it to have a strictly locally 1-to-1 image that is not a 2-to-1 retract.

Lemma 1 *If f is strictly locally 1-to-1 map from the continuum X into the continuum Y, and f is 1-to-1 on the closed subset A of X and 1-to-1 on $X\backslash A$, then Y is a 2-to-1 retract.*

Proof. Let $X_o = \{x \in X \mid f^{-1}(f(x)) \neq \{x\}\}$. Since f is locally 1-to-1 X_o is closed. Let $U = X\backslash(X_o \cap A)$, and U is clearly an open set. Since f is 1-to-1 on A, for each x in $X_o \cap A$ there is an $\hat{x}$ in U such that $f(x) = f(\hat{x})$. Therefore $f(U) = f(X)$. There cannot be three elements of X with the same image under f since f is 1-to-1 on A and on $X\backslash A$, and if there are two elements of X with the same image under f, then one of them is in $X_o \cap A$. Therefore f is 1-to-1 on U. It follows now from Theorem 1 that Y is a 2-to-1 retract.

Lemma 2 *If f maps the compactum X onto Y so that (1) f is strictly locally 1- to-1, (2) f is 1-to-1 on each proper subcontinuum of X, and (3) there is at least one 1-to-1 point p in X (meaning that no other point in X maps to $f(p)$), then f has a 1-to-1 cover.*

Proof. First, for each set K in X, define $\hat{K}$ to be the points in $X \setminus K$ that map the same under f as some point in K. Note that if K is closed, then so is $\hat{K}$. Since the set of 1-to-1 points in X is open, there is an open set U containing p that is contained in the set of 1-to-1 points. The components of $X \setminus U$ are components of a compactum, so if C is such a component and $\epsilon > 0$, then there is an open and closed set $V(C)$ in the ϵ-neighborhood of C that contains C; further, since f is locally 1-to-1 and 1-to-1 on each subcontinuum, we may assume that f is 1-to-1 on $V(C)$. $V(C)$ is open and closed in $X \setminus U$. Let $V_1, V_2, ..., V_n$ be a finite cover of $X \setminus U$ consisting of these $V(C)$ sets. Now, let

$$W = U \cup V_1 \cup (V_2 \setminus \hat{V_1}) \cup ... \cup (V_n \setminus (\hat{V_1} \cup \hat{V_2} \cup ... \cup \hat{V}_{(n-1)})).$$

Each $\hat{V_i}$ is closed, so the parenthetical sets are each open (in $X \setminus U$). Since $W \setminus U$ is open in $X \setminus U$, W is open in X. And f is 1-to-1 on W and maps W onto Y, so $f|W$ is a 1-to-1 cover of f.

Corollary 5 *If f is a strictly locally 1-to-1 map from a continuum X into a continuum Y, f is 1-to-1 on each proper subcontinuum of X, and there is at least one 1-to-1 point for f, then Y is a 2-to-1 retract.*

Lemma 3 *If f is a strictly locally 1-to-1 map from a decomposable continuum X into a continuum Y, and f is 1-to-1 on each proper subcontinuum of X, then Y is a 2-to-1 retract.*

Proof. Since X is decomposable, X is the union of two proper subcontinua, A and B; and since f is 1-to-1 on each proper subcontinuum of X, f restricted to each of A and B is 1-to-1. Thus every point of $A \cap B$ is a 1-to-1 point and the hypothesis of Corollary 5 is satisfied. Hence Y is a 2-to-1 retract of a continuum.

Theorem 2 *If f is a strictly locally 1-to-1 map from a hereditarily decomposable continuum X into a continuum Y, then Y is a 2-to-1 retract of a continuum.*

Proof. If X' is minimal with respect to being a subcontinuum of X on which f is not 1-to-1 then the conditions of the previous lemma are satisfied by the restriction of f to X'. So $f(X')$ is a 2-to-1 retract of a continuum by Lemma 3 and, since every continuum that contains a 2-to-1 retract is itself a 2-to-1 retract, $f(X)$ is also a 2-to-1 retract of a continuum.

Theorem 3 *The image of a strictly locally 1-to-1 map defined on an arc-like continuum is a 2-to-1 retract of a continuum.*

Proof. Assume X is an arc-like continuum, and f is a strictly locally 1- to-1 map with domain X. Since f is locally 1-to-1, there is a positive number ϵ such that if $f(x) = f(y)$ and $x \neq y$, then $d(x, y) > \epsilon$. Let g be an ϵ-map onto $[0, 1]$, and let $A = \{x \in X \mid \exists x' \neq x \ni f(x') = f(x) \text{ and } g(x') < g(x)\}$. It is easy to verify that A is closed, and that if $U = X \backslash A$, then f is 1-to-1 on U and $f(U) = f(X)$. Hence f has a 1-to-1 cover and its image must be a 2-to-1 retract of a continuum.

Lemma 4 *If the continuum X is the union of two continua A and B and every strictly locally 1-to-1 image of A and every strictly locally 1-to-1 image of B is a 2-to-1 retract of a continuum, then every strictly locally 1-to-1 image of X is a 2-to-1 retract of a continuum.*

Proof. A strictly locally 1-to-1 map with domain X is either strictly locally 1-to-1 on A, strictly locally 1-to-1 on B, or 1-to-1 on A and on $X \backslash A$. In each case $f(X)$ is a 2-to-1 retract of a continuum; in the latter case by Lemma 1 and in the first two cases because $f(X)$ contains a 2-to-1 retract of a continuum.

Theorem 4 *If f is a strictly locally 1-to-1 map defined on a continuum X that is a finite union of continua which are either arc-like or hereditarily decomposable, then $f(X)$ is a 2-to-1 retract of a continuum.*

Proof. Theorem 4 follows from Lemma 4, Theorem 2 and Theorem 3.

We would like to be able to replace arc-like with tree-like in Theorem 3. In Theorem 5 we come close, but there is an added assumption that the map is simple. We conjecture that this assumption is not necessary.

Lemma 5 *No tree-like continuum admits a non-trivial k-fold covering map.*

Proof. Every k-fold covering map is open and therefore, by a theorem of G.T. Whyburn [7, Theorem 7.5, p. 148], confluent. McLean [6] has shown that the confluent image of a tree-like continuum is itself a tree- like continuum and Maćkowiak [5] has shown that a local homeomorphism onto a tree-like continuum is a homeomorphism. Hence, any covering map defined on a tree-like continuum must be the trivial 1-to-1 covering map.

Theorem 5 *The image of a simple strictly locally 1-to-1 map defined on a treelike continuum is a 2-to-1 retract of a continuum.*

Proof. Suppose we have a simple strictly locally 1-to-1 map defined on a treelike continuum; then there is a restriction, say f, to a tree-like subcontinuum X of the domain such that f is strictly locally 1-to-1 and is 1-to-1 on each proper subcontinuum of X. So f cannot be a covering map by the previous lemma. Hence, since it is locally 1-to-1 it cannot be exactly 2- to-1; and so, since f is simple, there is a point in X at which f is 1-to-1. Thus, by Corollary 5, $f(X)$, and thus the original image space, is a 2- to-1 retract of a continuum.

Question 1 *Is the hypothesis that the map be simple necessary in Theorem 5?*

Corollary 6 *If f is a simple strictly locally 1-to-1 map defined on a continuum X that is a finite union of continua that are either tree-like or hereditarily decomposable, then $f(X)$ is a 2-to-1 retract.*

To find an example of a continuum that has a strictly locally 1-to-1 image that is not a 2-to-1 retract it is natural to think of a continuum that is hereditarily indecomposable with a locally 1-to-1 image that is also hereditarily indecomposable. That makes a 2-fold cover from the pseudo-circle onto itself

a natural choice. Note that the pseudo-circle is an example of a continuum that is a 2-to-1 image of a continuum but is not a 2-to-1 retract of a continuum. In the second example the domain and range are decomposable, but just barely so.

Example 1. A pseudo-circle is a hereditarily indecomposable, circularly chainable, separating plane continuum. It was shown in [2, Example 1] that there is a 2-fold cover, and therefore a strictly locally 1-to-1 map, from the pseudo-circle onto itself , and in [1, Theorem 5] that no hereditarily indecomposable continuum is a 2-to-1 retract of a continuum. The 2-fold cover is a simple strictly locally 1-to-1 map but every restriction of the 2-fold cover to a proper subcontinuum of the pseudo- circle is a homeomorphism.

Example 2. The continuum X is the union of two pseudo- circles, P_1 and P_2, joined at two points, and its image Y is the union of two pseudo-circles, Q_1 and Q_2, joined at one point. As mentioned in Example 1, there are 2-fold covers, g_1 and g_2, from P_1 onto Q_1, and from P_2 onto Q_2, respectively. Suppose a and b are points in P_1 such that $g_1(a) = g_1(b)$, and c and d are points in P_2 such that $g_2(c) = g_2(d)$. To form X, attach a in P_1 to c in P_2, and attach b in P_1 to d in P_2. To form Y, attach $g_1(a)$ in $Q_1 = g_1(P_1)$ to $g_2(c)$ in $Q_2 = g_2(P_2)$. Then the map $g_1 \cup g_2$ is a simple, strictly locally 1-to-1 map from X onto Y. Since the pseudo-circle is not a 2-to-1 retract, Y is not a 2-to-1 retract by Theorem 6 which is proven below.

4 When are decomposable continua 2-to-1 retracts?

Suppose $X = A \cup B$ is a decomposable continuum, and A and B are proper subcontinua. When is X a 2-to-1 retract? If $A \cap B$ is not connected then X is not unicoherent and we know from [4] that X is a 2-to-1 retract. If either A or B is a 2-to-1 retract, then we know from [3] that X is a 2-to-1 retract. But, are these conditions necessary? Yes. We show in Theorem 6 that if A and B both fail to be 2-to-1 retracts and if their intersection is connected, then X cannot be a 2-to-1 retract.

Lemma 6 *If X is a 2-to-1 retract, and K is a subcontinuum of X, then*

there is a component C of $X\backslash K$ such that $C \cup K$ is a 2- to-1 retract.

Proof. Let $r : Z \to X$ be a 2-to-1 retraction from a continuum Z onto X. If $r^{-1}(K)$ is connected, then K is a 2-to-1 retract, so the conclusion is true for any component of $X\backslash K$. So, assume that $r^{-1}(K)$ is not connected. Then $r^{-1}(K)$ is contained in $D \cup E$, where D and E are disjoint open sets intersecting $r^{-1}(K)$. Without loss of generality, we will assume that K is in D. Let K' be a copy of K disjoint from Z. For each point x in $r^{-1}(K) \cap D$, identify x with $r(x)$, and for each point x in $r^{-1}(K) \cap E$, identify x with the copy of $r(x)$ in K'. Call this new continuum Z'. We then have a 2-to-1 retraction R from Z' onto X for which $R^{-1}(K)$ has exactly two components, K and K'. There is a component C' of $Z'\backslash(X \cup K')$ whose closure intersects both X and K' since Z' is connected. Let C denote the component of $X\backslash K$ that contains $R(C')$. Some point x of X is the limit of a sequence S of points of C' and x must also be the limit of the sequence $R(S)$. Hence $T = K \cup C \cup C' \cup K'$ is connected. Every component of $Z'\backslash(X \cup K')$ either maps into C or its image misses C, and the closure of each component of $Z'\backslash(X \cup K')$ intersects either K' or X. Suppose such a component V maps into C . If its closure intersects K', then $V \cup K'$ is connected and if its closure intersects X then its closure intersects C by the same argument that the closure of C' intersects C, so $V \cup C$ is connected. Hence, all of the components of $Z'\backslash(X \cup K')$ that map into C can be added to T, getting a connected set that is equal to $R^{-1}(K \cup C)$. Thus $K \cup C$ is a 2-to-1 retract.

Theorem 6 *Suppose $X = A \cup B$ is a decomposable continuum and each of A and B is a proper subcontinuum. Then X is a 2-to-1 retract iff one of the following is true:*

- *A is a 2-to-1 retract, or*
- *B is a 2-to-1 retract, or*
- *$A \cup B$ is not connected.*

Proof. The sufficiency of each of the three conditions is discussed at the beginning of Section 4. For the converse, assume $X = A \cup B$ is a 2-to-1 retract of a continuum, and that $K = A \cap B$ is connected. Then, by Lemma 6, there is a component C of $X\backslash K$ such that $C \cup K$ is a 2-to- 1 retract. But C must either be a subset of A or of B. If $C \subset A$, then $C \cup K$ is a 2-to-1

retract in A which implies that A itself is a 2- to-1 retract. Thus, one of the three conditions has to hold.

5 2-to-1 retract implies k-to-1 retract.

Information we have on which continua are 2-to-1 retracts helps with the study of which continua are k-to-1 retracts, for other positive integers k, by way of the corollary below.

Theorem 7 *Suppose Y is a k-to-1 retract of a continuum. Then, for each positive integer n, Y is a $(1 + (k-1) \times n)$-to-1 retract of a continuum.*

Proof. Suppose X is a continuum and $r : X \to Y$ is a k-to-1 retraction onto Y. Let n be a positive integer. Define the map $g_i : X \to X \times \prod_{j=1}^{n} [0, \infty)$ for $1 \leq i < n$ by letting $g_i(x)$ be the point in $X \times \prod_{j=1}^{n} [0, \infty)$ with first coordinate x, with $i + 1$ coordinate $d(x, Y)$, and with all other coordinates zero. Let $Y' = Y \times \prod_{j=1}^{n} \{0\}$. Let $Z = Y' \cup (\bigcup_{i=1}^{n} g_i(X))$. Since $g_i(X)$ intersects Y' for each i, Z is a continuum. The $(1 + ((k-1) \times n))$-to-one retraction $r^* : Z \to Y'$ is defined by $r^*((x, t_1, t_2, ..., t_n)) = (r(x), 0, 0, ..., 0)$. The conclusion of the theorem follows because Y is homeomorphic to Y'.

Corollary 7 *If a continuum Y is a 2-to-1 retract of a continuum, then Y is a k-to-1 retract of a continuum, for each $k > 2$.*

6 Glossary

- **Arclike.** A continuum is *arclike* if for each $\epsilon > 0$, there is an ϵ-map from the continuum onto an arc.
- **Confluent Map**. A map is *confluent* if each component of the inverse of any continuum C in the image is mapped onto C.
- **Continuum.** A topological space is a *continuum* if it is connected, compact, and metric.
- **Covering Map.** A map defined on a continuum is a *covering map* if it is k-to-1 for some positive integer k, open, and locally 1-to-1.

- **Indecomposable Continuum.** A continuum is *indecomposable* if it is not the union of two proper subcontinua.
- **Local Cut Continuum and Local Cut Point** A subcontinuum K of a continuum Y is a local cut continuum if $Y \backslash K$ is connected but there is an open set U containing K such that $U \backslash K = A \cup B$, two nonempty separated sets, and K contains both a limit point of A and a limit point of B. If K consists of a single point then that point is called a local cut point.
- **Map.** A *map* is a continuous function.
- **1-to-1 cover** A *1-to-1 cover* of a map f with domain X is a restriction of f to an open proper subset U of X such that f is 1-to-1 on U and $f(U) = f(X)$.
- **Simple Map** A map is *simple* if the cardinality of each point inverse is either one or two.
- **Strictly locally 1-to-1.** A *strictly locally 1-to-1 map* is a map which is locally 1-to-1 but not a homeomorphism.
- **Treelike.** A continuum is *treelike* is for each $\epsilon > 0$,

  there is an ϵ-map from the continuum onto a tree (an acyclic graph).
- **2-to-1.** A function is *2-to-1* if the preimage of each point in the image has exactly two points.
- **2-to-1 retract** A continuum Y is a *2-to-1 retract* if there is a continuum X and a retraction r from X onto a subcontinuum of X that is homeomorphic to Y.
- **Unicoherent Continuum** A continuum is *unicoherent* if it is not the union of two subcontinua whose intersection is not connected.

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